Name:	
Student ID:	

Final Examination

Phys210: Mathematical Methods in Physics II

2025/06/12

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selections in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ______________________________

This exam has a total of 3 questions, some of which may be for bonus points. You can obtain a maximum grade of 34+0 from this examination.

Question	Points	Score
1	6	
2	$10\frac{1}{2}$	

Question	Points	Score
3	$17\frac{1}{2}$	
Total:	34	

1 Notations & Conventions

This section contains various useful definitions to refer while solving the problems. Note that it might contain additional information not covered in class, so please do not panic: the questions do not necessarily refer to *everything* in this section.

Set for our purposes is a collection of elements such that any given object either belongs or not to the chosen collection.

Boolean (denoted $\mathbb{B}$) is a set with only two elements, usually chosen as True and False. Engineers also prefer 0 & 1.

Predicate is any function whose codomain is the set `Boolean`. For instance, $f : \mathbb{R} \to \mathbb{B}$ with

$$f(x) = (x > 2) \tag{1}$$

is a predicate with $f(3) = \mathsf{True}$ and $f(1) = \mathsf{False}$.

Set comprehension is a method to generate subsets from a big set via a predicate. The usual convention is $\{x \in S \mid P(x)\}$ which is a set whose elements are elements of S for which $P(x)$ is true. For instance, $\{x \in \mathbb{Z} \mid x^2 > 100\}$ is the set of integers whose square is greater than 100.

The non-negative integer power of an object A (denoted A^n) is defined recursively as

$$A^0 = \mathbb{I}\ , \quad A^n = A \cdot A^{n-1} \quad \forall n \geq 1 \tag{2}$$

with respect to the operation $\cdot$ (such as matrix multiplication or differentiation) and its identity object $\mathbb{I}$.

Exponentiation of an object A (denoted e^A) is

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{3}$$

where A^n is the $n-$th power of the object A.

Logarithm of an object A (denoted $\log A$) is defined as the inverse of the exponentiation. For objects for which the exponentiation is not a monomorphism (such as complex numbers), logarithm is a *relation* (also called multi-valued function). Conventionally, one imposes restrictions on the domain to ensure that logarithm acts as a function; for instance, for a complex number $z = re^{i\theta} \in \mathbb{C}$ with $(r, \theta) \in (\mathbb{R}^+, \mathbb{R})$, we can define $\log z = i\theta_p + \log r$ where $0 \leq \theta_p < 2\pi$ is called *the principal value of* θ.

The generalized power of an object A (denoted A^α) is defined as

$$A^\alpha = e^{\alpha \log A} \tag{4}$$

If exponentiation is not a monomorphism when acting on the domain of A, A^α is not a function but a relation *unless* a principle domain is selected (similar to the logarithm).

Generalized exponentiation of an object A (denoted α^A) is defined as

$$\alpha^A = e^{A \log \alpha} \tag{5}$$

Depending on the available values for $\log \alpha$, α^A may mean multiple different functions. However, each one is *still* a proper function, not a multi-valued function.

Trigonometric functions $\cos,\ \sin,\ \tan,\ \cot,\ \csc\ , \sec$ are defined in terms of the exponential via the equations

$$e^{\pm iA} = \cos(A) \pm i \sin(A)\ ,\ \tan(A) = \frac{1}{\cot(A)} = \frac{\sin(A)}{\cos(A)} \tag{6}$$

$$\csc(A) \sin(A) = 1\ ,\ \sec(A) \cos(A) = 1 \tag{7}$$

Hyperbolic functions $\cosh,\ \sinh,\ \tanh,\ \coth,\ \mathrm{csch}, \mathrm{sech}$ are defined in terms of the exponential via equations

$$e^{\pm A} = \cosh(A) \pm \sinh(A), \tanh(A) = \frac{1}{\coth(A)} = \frac{\sinh(A)}{\cosh(A)} \tag{8}$$

$$\mathrm{csch}(A) \sinh(A) = 1\ ,\ \mathrm{sech}(A) \cosh(A) = 1 \tag{9}$$

Inverse Trigonometric/Hyperbolic functions are denoted with an *arc* prefix in their naming, i.e. $\arcsin(x) := \sin^{-1}(x)$. Like logarithm, these objects are *relations* (not functions) unless their domain is restricted.

The Kronecker symbol (Kronecker-delta) is defined

$$\delta : \{\mathbb{Z}, \mathbb{Z}\} \to \mathbb{Z} \tag{10}$$

$$\delta = \{i, j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{11}$$

The Dirac-delta generalized function δ is (for all practical purposes of a Physicist) defined via the relation

$$\int_{\mathcal{A}} f(y) \delta(x - y) dy = \begin{cases} f(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

A useful representation of this generalized function is

$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \tag{13}$$

which can also be understood as Fourier transform of 1.

Heaviside generalized function θ is (for all practical purposes of a Physicist) defined via the relations

$$\int_a^b \theta(x) f(x) dx = \begin{cases} \int_a^b f(x) dx & \text{if } a \geq 0 \\ \int_0^b f(x) dx & \text{if } a < 0 \end{cases} \tag{14}$$

This definition implies that $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$; however, it *does not fix* $f(0)$. We choose *the convention* $f(0) = 1/2$; this ensures

$$\mathrm{sgn}(x) = 2\theta(x) - 1 = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases} \tag{15}$$

A particular permutation of n objects is denoted as $(i_1 i_2 \ldots i_n)$ where $i_1 \neq i_2 \neq \cdots \neq i_n \in \{1, \ldots, n\}$. A permutation $(i_1 \ldots i_n)$ is said to be an even (odd) permutation of $(k_1 \ldots k_n)$ if the two are identical after the permutation of an even (odd) number of adjacent indices. For example, (2431) is an even permutation of (2143) and an odd one of (2134).

Levi-Civita symbol ϵ is defined as

$$\epsilon : \{\mathbb{Z}^+, \dots, \mathbb{Z}^+\} \to \mathbb{Z} \tag{16}$$

$$\epsilon = \{a_1, \dots, a_n\} \to \begin{cases} 1 & \text{if } (a_1 a_2 \dots a_n) \text{ is an even permutation of } (12\dots n) \\ -1 & \text{if } (a_1 a_2 \dots a_n) \text{ is an odd permutation of } (12\dots n) \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

The determinant function (denoted det) is defined

$$\det : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{18}$$

$$\det = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_{i_1,\dots,i_n} \epsilon_{i_1\dots i_n} a_{1i_1} \dots a_{ni_n} \tag{19}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

The adjugate function (denoted adj) is defined as

$$\mathrm{adj} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{20}$$

$$\mathrm{adj} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \tag{21}$$

for

$$b_{k_n i_n} = \sum_{\substack{i_1,\dots,i_{n-1} \\ k_1,\dots,k_{n-1}}} \frac{\epsilon_{i_1\dots i_n} \epsilon_{k_1\dots k_n} a_{i_1 k_1} \dots a_{i_{n-1} k_{n-1}}}{(n-1)!} \tag{22}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

Inverse of an object A is denoted as A^{-1} and is defined with respect to an operation "$\cdot$" and its identity element $\mathbb{I}$ via the the equations $A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{I}$. If "$\cdot$" is matrix multiplication, then

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det A} \tag{23}$$

The trace function (denoted tr) is defined as

$$\mathrm{tr} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{24}$$

$$\mathrm{tr} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_i a_{ii} \tag{25}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

Wronskian matrix of a set of functions $\{f_1(x), \dots, f_n(x)\}$ is defined as a square matrix where the first row is the set of the functions and the i−th row is $(i-1)$−th derivative of the functions for all $n \geq i \geq 2$.

A complex number z is (for all practical purposes of a Physicist) a pair of two real numbers (x, y) where one can construct z via $z = x + iy$ (i is called *the imaginary unit* with the property $i^2 = -1$); conversely, one can extract x and y via $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

Complex conjugation (denoted $*$) is a function defined to act on complex numbers as

$$* : \mathbb{C} \to \mathbb{C} \tag{26}$$

$$* = z \to (z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z)) \tag{27}$$

Matrix transpose (denoted T) is a function defined

$$T : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{28}$$

$$T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \tag{29}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

Hermitian conjugation (also called *conjugate transpose*, *adjoint*, or *dagger*) is a function defined as

$$\dagger : \mathfrak{M}_{n\times n}(\mathbb{C}) \to \mathfrak{M}_{n\times n}(\mathbb{C}) \tag{30}$$

$$\dagger = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a^*_{11} & a^*_{21} & \dots & a^*_{n1} \\ a^*_{12} & a^*_{22} & \dots & a^*_{n2} \\ \dots \\ a^*_{1n} & a^*_{2n} & \dots & a^*_{nn} \end{pmatrix} \tag{31}$$

Characteristic polynomial of any square matrix A is

$$\det (A - \lambda_i \mathbb{I}) = 0 \tag{32}$$

Laplace transform (denoted $\mathcal{L}$) is an integral transform which converts a function $f : \mathbb{R} \to \mathbb{R}$ into another function $\hat{f} = \mathcal{L}(f)$ such that

$$\hat{f} : \mathbb{C} \to \mathbb{C} , \qquad \hat{f}(s) = \int_0^\infty f(x) e^{-xs} dx \tag{33}$$

For *meromorphic* $\hat{f}$ (i.e. $\frac{\text{polynomial}}{\text{polynomial}}$), the inverse is computed by rewriting $\hat{f}(s)$ as a sum $\sum_i a_i (s + r_i)^{-n_i - 1}$ which is clearly (for some $c_{k,\ell}$) the Laplace transform of $f(x) = \sum_i e^{-r_i x} \left(c_{i,1} + c_{i,2} x + \dots c_{i,n_i} x^{n_i}\right)$. Formally,

$$f : \mathbb{R} \to \mathbb{R} , \qquad f(x) = \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{f}(s) e^{xs} \frac{ds}{2\pi i} \tag{34}$$

where the *contour integral* in the complex plane is chosen appropriately based on the convergence.

Convolution of two functions f and g (denote $f * g$) is the operation that becomes multiplication in the Laplace domain, i.e. $\mathcal{L}(f * g) \equiv \mathcal{L}(f)\mathcal{L}(g)$; equivalently,

$$(f * g)(x) = \int_0^x f(y) g(x - y) dy \tag{35}$$

Fourier transforms are widely-used integral transformations, the simplest examples of harmonic analysis, and can be defined with any self-consistent convention. We choose

$$f : \mathbb{R} \to \mathbb{C} , \qquad f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}(k) \tag{36}$$

$$\hat{f} : \mathbb{R} \to \mathbb{C} , \qquad \hat{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \tag{37}$$

— — —

$$f : [a, a+T] \to \mathbb{C} , \qquad f(x) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{T} x} \hat{f}(n) \tag{38}$$

$$\hat{f} : \mathbb{Z} \to \mathbb{C} , \qquad \hat{f}(n) = \int_a^{a+T} dx e^{-i\frac{2\pi n}{T} x} f(x) \tag{39}$$

— — —

$$f : \mathbb{Z} \to \mathbb{C}\,, \qquad f(n) = \frac{1}{T}\int_a^{a+T} dx e^{i\frac{2\pi n}{T}k}\hat{f}(k) \tag{40}$$

$$\hat{f} : [a, a+T] \to \mathbb{C}\,, \qquad \hat{f}(k) = \sum_{n=-\infty}^{\infty} e^{-i\frac{2\pi n}{T}k} f(n) \tag{41}$$

— — —

$$f : \mathbb{Z}_N \to \mathbb{Z}_N\,, \qquad f(n) = \frac{1}{N}\sum_{m=0}^{N-1} e^{i\frac{2\pi nm}{N}}\hat{f}(m) \tag{42}$$

$$\hat{f} : \mathbb{Z}_N \to \mathbb{Z}_N\,, \qquad \hat{f}(m) = \sum_{n=0}^{N-1} e^{-i\frac{2\pi nm}{N}} f(n) \tag{43}$$

where (36), (38), (40), and (42) are called *Fourier Transform*, *Fourier Series*, *Discrete-time Fourier Transform*, and *Discrete Fourier Series* respectively while the rest are their inverses. We will stick to this naming but please be reminded that different communities (engineering, math, physics, etc.) use different naming conventions in general.

Rectangle function (also called *unit pulse*, *window / gate function*) is defined in terms of Heaviside function as

$$\mathrm{rect}(x) = \theta\left(x + \frac{1}{2}\right) - \theta\left(x - \frac{1}{2}\right) \tag{44}$$

Sine cardinal function (denoted sinc) is defined as $\mathrm{sinc}(0) = 1$ and $\mathrm{sinc}(x) = \sin(x)/x$ for $x \neq 0$. It is the Fourier transform of rect function, i.e. $\mathrm{rect}(x) \leftrightarrow \mathrm{sinc}(k/2)$.

"Even part of" and "odd part of" (denoted E and O) are higher order functions defined as

$$E : (\mathcal{A} \to \mathcal{A}) \to (\mathcal{A} \to \mathcal{A}) \tag{45}$$

$$E = (x \to f(x)) \to \left(x \to f_E(x) = \frac{f(x) + f(-x)}{2}\right) \tag{46}$$

$$O : (\mathcal{A} \to \mathcal{A}) \to (\mathcal{A} \to \mathcal{A}) \tag{47}$$

$$O = (x \to f(x)) \to \left(x \to f_E(x) = \frac{f(x) - f(-x)}{2}\right) \tag{48}$$

with which any single-argument function satisfies $f = E \cdot f + O \cdot f$ (more commonly written $f(x) = f_E(x) + f_O(x)$).

Inner product between two functions f and g shall be denoted in this exam for $\mathcal{A} \subseteq \mathbb{R}$ as $\langle f, g\rangle_\omega^{\mathcal{A}}$:

$$\langle,\rangle_\omega^{\mathcal{A}} : (\mathcal{A} \to \mathbb{C}, \mathcal{A} \to \mathbb{C}) \to \mathbb{C} \tag{49}$$

$$\langle f, g\rangle_\omega^{\mathcal{A}} = \int_A \left(f(x)\right)^* g(x)\omega(x)dx \tag{50}$$

Group is defined as a pair (S, o) where S : `Set` and where $o : (S, S) \to S$ for which the following are true:

1. $(\exists e \in S)(\forall s \in S)\ o(e, s) = o(s, e) = s$
2. $(\forall s \in S)\ o(s, i(s)) = o(i(s), s) = e$
3. $(\forall a, b, c \in S)\ o(a, o(b, c)) = o(o(a, b), c)$

for a unique function $i : S \to S$.

Ring is defined as a triplet $(S, +, \cdot)$ where S : `Set`, and $+, \cdot : (S, S) \to S$ for which the following are true:

1. $(S, +)$: `Commutative Group`
2. $(\forall a, b, c \in S)\ a \cdot (b + c) = a \cdot b + a \cdot c$
3. $(\forall a, b, c \in S)\ (b + c) \cdot a = b \cdot a + c \cdot a$

Skew field is defined as a triplet $(S, +, \cdot)$ where S : `Set`, and $+, \cdot : (S, S) \to S$ for which following are true:

1. $(S, +, \cdot)$: `Ring`
2. $(S \backslash \{0\}, \cdot)$: `Group`

where 0 denotes the identity element with respect to +.

Field is defined as a triplet $(S, +, \cdot)$ where S : `Set`, and $+, \cdot : (S, S) \to S$ for which the following are true:

1. $(S, +, \cdot)$: `Ring`
2. $(S \backslash \{0\}, \cdot)$: `Commutative Group`

where 0 denotes the identity element with respect to +.

Linear space (also called *vector space*) over a field $F = (S, +, \cdot)$ shall be denoted as $V(F)$ and is defined as a triplet $(V, \oplus, \odot)$ with V : `Set`, $\oplus : (V, V) \to V$, and $\odot : (S, V) \to V$ for which the following are true:

1. $(V, \oplus)$: `Commutative Group`
2. $(\forall v \in V)\ 1 \odot v = v$ (1 is the identity element of $\cdot$)
3. $(\forall v \in V)(\forall s \in S)\ s \odot v \in V$
4. $(\forall v \in V)(\forall a, b \in S)\ (a \cdot b) \odot v = a \odot (b \odot v)$
5. $(\forall v \in V)(\forall a, b \in S)\ (a + b) \odot v = (a \odot v) \oplus (b \odot v)$
6. $(\forall v, w \in V)(\forall s \in S)\ s \odot (v \oplus w) = (s \odot v) \oplus (s \odot w)$

The elements of the set S (V) are called *scalars* (*vectors*).

Linear algebra (also called *vector algebra*) over a field $F = (S, +, \cdot)$ shall be denoted as $L(F)$ and is defined as a quadruple $(V, \oplus, \odot, \otimes)$ for which the following are true:

1. $(V, \oplus, \odot)$: `Linear Space`
2. $(\forall x, y, z \in V)\ x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
3. $(\forall x, y, z \in V)\ (x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$
4. $(\forall x, y \in V)(\forall a, b \in S)\ (a \odot x) \otimes (b \odot y) = (a \cdot b) \odot (x \otimes y)$

Lie algebra is a linear algebra $(V, \oplus, \odot, \otimes)$ with the additional condition that $(\forall x, y \in V)\ x \otimes y = -y \otimes x$.

Commutator is a higher order function which takes two functions $f, g : \mathcal{A} \to \mathcal{A}$ for any type $\mathcal{A}$, and gives a new function $[f, g] : \mathcal{A} \to \mathcal{A}$ by cascading their action. It is defined on an object $x \in \mathcal{A}$ as $[f, g](x) = f(g(x)) - g(f(x))$.

Basis B of a vector space V is $(B \supset V)$: `Set` such that

1. $(\forall k \in \{1, 2, \dim B\})(\forall e_1, \ldots, e_k \in B)(\forall c_1, \ldots, c_k \in S)\ [c_1 = \cdots = c_k = 0] \vee [c_1 e_1 + \cdots + c_k e_k \neq 0]$
2. $(\forall v \in V)(\exists! a_1, \ldots, a_{\dim B} \in S) v = a_1 e_1 + \ldots + a_{\dim B} e_{\dim B}$

Normed vector space over a field F is a vector space $V(F)$ over which a function norm : $V \to \mathbb{R}$ exists with the notation norm $= x \to \|x\|$, for which following are true:

1. $(\forall v \in V)[\|v\| \neq 0] \vee [v = 0]$
2. $(\forall v \in V)(\forall s \in F)\|s \odot v\| = |s|.\|v\|$
3. $(\forall v, w \in V)\|v \oplus w\| \leq \|v\| + \|w\|$

Inner product vector space over a field F is a vector space $V(F)$ over which a function $\langle\rangle : (V, V) \to \mathbb{C}$ exists for which following statements are true:

1. $(\forall v, w \in V)\ \langle v, w\rangle = \langle w, v\rangle^*$
2. $(\forall u, v, w \in V)(\forall a, b \in F)\,\langle au + bv, w\rangle = a\,\langle u, w\rangle + b\,\langle v, w\rangle$
3. $(\forall v \in V\backslash\{0\})\ \langle v, v\rangle > 0$
4. $\langle 0, 0\rangle = 0$

Dual of a vector space $V(F)$ is a vector space denoted as $V^*(F)$ whose elements are linear functions from the vector space $V(F)$ to the underlying field F.

Type (r, s) tensor on a vector space V is an element of vector space $\underbrace{V \otimes V \otimes \cdots V}_{r} \otimes \underbrace{V^* \otimes V^* \cdots \otimes V^*}_{s}$ where $\otimes$ is an associative bilinear map.

Tensor algebra $T(V)$ over a vector space V is the direct sum of all possible (r, s) tensor spaces, with the $\otimes$ being the natural product between different tensors.

Multivector (also called k−vector) is an element of the vector space whose elements are constructed via the associative antisymmetric *wedge product* $\wedge$ of the underlying vectors; e.g. $u \wedge v$ is a 2-vector if u and v are vectors.

Exterior algebra $\Lambda(V)$ over a vector space V is the direct sum of all possible multivectors, with wedge product $\wedge$ being the natural product between multivectors.

Covariant & Contravariant indices in our conventions refer to *downstairs* and *upstairs* indices of a tensor's components, hence are multiplied with basis vectors of V^* and V to yield the full tensor, e.g. $T = T^{ij}{}_{k}\ e_i ⓡ e_j ⓡ e^k$ with $T^{ij}{}_{k}$ having one covariant and two contravariant indices where ⓡ is the associative binary operation of the algebra, e.g. $\otimes, \wedge, \dots$.

Contraction is the action of applying a dual vector ($V \to S$) to a vector (V), hence reducing a (r, s)−tensor to a $(r-1, s-1)$−tensor. In an orthonormal basis with $e^i(e_k) = \delta^i_k$ (such as Cartesian coordinates), this amounts to summing over a covariant and a contravariant indices.

Manifold is (for our purposes) any space that resembles $\mathbb{R}^d$ near its every point, for instance the sphere S^2.

(Co)tangent space to a manifold M at a point x is $\mathbb{R}^d$ centered at x and is denoted as T_xM (T^*_xM). The (co)tangent space is inhabited by the (co)vectors at $x \in M$, with the basis vectors usually chosen as $\frac{\partial}{\partial x^i}$ (dx_i).

(Co)tangent bundle is *disjoint union* of (co)tangent spaces of a manifold M, and is denoted as TM (T^*M).

Musical isomorphism between a tangent and cotangent bundle is initiated with two functions: $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$, hence for instance $(x^i e_i)^\flat = (x_i e^i)$, and $(x_i e^i)^\sharp = (x^i e_i)$

Field in Physicist terminology broadly refers to any map from a manifold M to *something* ($\mathbb{R}$, TM, …). The field is *named* appropriately depending on the output: scalar field ($M \to \mathbb{R}$), vector field ($M \to TM$), tensor field ($M \to (TM \otimes TM \otimes T^*M \otimes \cdots)$), and so on.

Differential forms (or forms for short) are functions that takes a point x from a Manifold M and yields a multi(co)vector from the exterior algebra of the (co)tangent space of M at x, e.g. $\omega = (x, y) \to dx + y dy$.

Hodge dual of a multivector or a form α is denoted as $\star\alpha$, and their components in $\mathbb{R}^d$ are related to one another for $\alpha = \alpha_{i_1\dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$ and $\star\alpha = (\star\alpha)_{i_{k+1}\dots i_d} e^{i_{k+1}} \wedge \cdots \wedge e^{i_d}$ as

$$(\star\alpha)_{i_{k+1}\dots i_d} = \frac{1}{(d-k)!}\alpha_{i_1\dots i_k}\epsilon^{i_1\dots i_k \ell_{k+1}\dots\ell_d}\delta_{i_{k+1}\ell_{k+1}}\cdots\delta_{i_d\ell_d}$$

Exterior derivative takes a p−form ω to $p+1$ form $d\omega$; with the basis vectors $\{dx^i\}$, it reads as

$$\omega = \omega_{i_1\dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{51}$$

$$d\omega = \frac{\partial \omega_{i_1\dots i_p}}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{52}$$

Gradient (denoted grad) is a function `Scalar Field` → `Vector Field`, defined as grad $= f \to (df)^\sharp$. ∇f is also used as a notation for grad(f). In Cartesian coordinates,

$$\begin{aligned}\text{grad} =& \big((x_1, \dots, x_d) \to f(x_1, \dots, x_d)\big) \\ &\to \left((x_1, \dots, x_d) \to \frac{\partial f(x_1, \dots, x_d)}{\partial x_i}\hat{x}_i\right)\end{aligned} \tag{53}$$

Divergence (denoted div) is a function `Vector Field` → `Scalar Field`, defined as grad $= v \to \big(\star d \star v^\flat\big)$. $\nabla \cdot v$ is also used as a notation for div(v). In Cartesian coordinates,

$$\begin{aligned}\text{div} =& \big((x_1, \dots, x_d) \to v^i(x_1, \dots, x_d)\hat{x}_i\big) \\ &\to \left((x_1, \dots, x_d) \to \frac{\partial v^i(x_1, \dots, x_d)}{\partial x^i}\right)\end{aligned} \tag{54}$$

Curl (denoted curl) is a function `Vector Field` → $(d-2)-$`Vector Field`, defined as curl $= v \to \big(\star dv^\flat\big)^\sharp$. In $d = 3$, $\nabla \times v$ is also used as a notation for curl(v); in Cartesian coordinates,

$$\nabla \times v = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} \tag{55}$$

Laplacian (for our purposes) is a function `Tensor Field` → `Tensor Field`, denoted as Δ (sometimes also as ∇^2), and is defined in Cartesian coordinates as

$$\begin{aligned}R &: TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M \\ R &= R^{i_1\dots i_r}{}_{k_1\dots k_s}\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{k_1} \otimes \cdots \otimes dx^{k_s} \\ \Delta R &: TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M \\ \Delta R &= \frac{\partial^2 R^{i_1\dots i_r}{}_{k_1\dots k_s}}{\partial x^m \partial x_m}\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{k_1} \otimes \cdots \otimes dx^{k_s}\end{aligned} \tag{56}$$

For instance, in three dimensional Euclidean space

$$\Delta\vec{V}(x, y, z) = \frac{\partial^2 V_x(x, y, z)}{\partial x^2}\hat{x} + \frac{\partial^2 V_y(x, y, z)}{\partial y^2}\hat{y} + \frac{\partial^2 V_z(x, y, z)}{\partial z^2}\hat{z} \tag{57}$$

when acting on a vector field $\vec{V}(x, y, z)$.

Differentiation identities follow from the following three properties of the exterior algebra: $d^2 = 0$, $(a^\flat)^\sharp = (a^\sharp)^\flat = a$, and $\star\star a = a$. These imply

$$\text{curl}(\text{grad}(f)) = \text{div}(\text{curl}(v)) = 0 \tag{58}$$

for any f and v, which in three dimensional Euclidean spaces can also be denoted as

$$\nabla \times \nabla f = \nabla \cdot \nabla \times \mathbf{v} = 0 \tag{59}$$

Product rule for differentiation for vector fields follow from Leibniz rule for the exterior derivative, i.e.

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta \quad (60)$$

for the $p-$form ω. For $3d$ Euclidean spaces, this implies

$$\begin{aligned}\nabla\cdot(a\mathbf{b}) &= \nabla a\cdot\mathbf{b} + a\nabla\cdot\mathbf{b}\,, \quad \nabla\times(a\mathbf{b}) = \nabla a\times\mathbf{b} + a\nabla\times\mathbf{b}\\ \nabla(\mathbf{a}\cdot\mathbf{b}) &= (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})\\ \nabla\cdot(\mathbf{a}\times\mathbf{b}) &= \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})\\ \nabla\times(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} \quad (61)\end{aligned}$$

A harmonic tensor field R (a harmonic function being a special case as a harmonic type $(0,0)$ tensor field) is an element of the kernel of the Laplacian, i.e. $\Delta R = 0$.

Helmholtz decomposition of a $3d$ vector field $\mathbf{E}$ is a way of rewriting it in terms of its *scalar potential* Φ (related to the divergence of vector field) and its *vector potential* $\mathbf{V}$ (related to the curl of vector field): $\mathbf{E} = \mathbf{c} - \nabla\Phi + \nabla\times\mathbf{V}$ for a constant vector $\mathbf{c}$ where

$$\begin{aligned}\Phi(r) &= \frac{1}{4\pi}\int_{\text{manifold}} \frac{\nabla'\cdot\mathbf{E}(r')}{|r-r'|}dV' - \frac{1}{4\pi}\oint_{\text{boundary}} \frac{\hat{\mathbf{n}}'\cdot\mathbf{E}(r')}{|r-r'|}dS'\\ \mathbf{V}(r) &= \frac{1}{4\pi}\int_{\text{manifold}} \frac{\nabla'\times\mathbf{E}(r')}{|r-r'|}dV' - \frac{1}{4\pi}\oint_{\text{boundary}} \frac{\hat{\mathbf{n}}'\times\mathbf{E}(r')}{|r-r'|}dS' \end{aligned} \quad (62)$$

Vectors in $3d$ Euclidean Space satisfy following formula

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \quad (63)\\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \quad (64)\\ (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \quad (65)\end{aligned}$$

Arc-length is the length of a curve (denoted by s), which satisfies $s = \int_{t_0}^{t} \left|\frac{d\mathbf{x}(t')}{dt'}\right| dt'$. In this equation, $\mathbf{x}(t)$ is the position of a point on the curve, t is the parametrization parameter, and t_0 is the value of t at the starting point of the curve. The arc-length itself can be used to parametrize the curve.

Tangent vector to a curve in the arc-length parametrization is the function $\mathbf{t}(s) = \frac{d\mathbf{x}(s)}{ds}$. It has unit norm, and can be likened to the ratio velocity per speed.

Curvature of a curve κ is a function of the arc-length whose value is $\kappa(s) = \left|\frac{d\mathbf{t}(s)}{ds}\right|$.

Principle normal of a curve $\mathbf{n}$ is a function of the arc-length whose value is $\mathbf{n}(s) = \frac{1}{\kappa(s)}\frac{d\mathbf{t}(s)}{ds}$. It has unit norm, and can be likened to the acceleration unit vector.

Binormal vector of a curve $\mathbf{b}$ is a function of the arc-length whose value is $\mathbf{b}(s) = \mathbf{t}(s)\times\mathbf{n}(s)$ ($|\mathbf{b}(s)| = 1$).

Torsion of a curve τ is a function of the arc-length whose value is $\tau(s) = -\mathbf{n}(s)\cdot\frac{d\mathbf{b}(s)}{ds}$.

The Frenet-Serret equations is a closed system of equations which completely determine the properties of a curve as a function of the curvature and torsion functions:

$$\begin{aligned}\frac{d\mathbf{t}(s)}{ds} &= \kappa(s)\mathbf{n}(s)\,, \quad \frac{d\mathbf{b}(s)}{ds} = -\tau(s)\mathbf{n}(s)\,,\\ \frac{d\mathbf{n}(s)}{ds} &= \tau(s)\mathbf{b}(s) - \kappa(s)\mathbf{t}(s)\end{aligned} \quad (66)$$

Generalized Stokes theorem equates the integration of a $p-$form ω over the boundary of a manifold ∂M to the integration of the exterior derivative of the $p-$form $d\omega$ over the manifold M: $\int_{\partial M}\omega = \int_M d\omega$.

Integral theorems are special cases of the generalized Stokes theorem. For a volume $\mathbf{V} \in \mathbb{R}^3$, a surface $\mathbf{S} \in \mathbb{R}^3$, a curve $\gamma \in \mathbb{R}^3$, and a region $\mathbf{D} \in \mathbb{R}^2$ (and for the notation ∂A being boundary of A), we have

$$\begin{aligned}&\int_{\mathbf{V}} \nabla\cdot\mathbf{F}dV = \oint_{\partial\mathbf{V}} \mathbf{F}\cdot d\mathbf{S}\,, \quad \int_{\mathbf{S}} (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \oint_{\partial\mathbf{S}} \mathbf{F}\cdot d\mathbf{\Gamma}\\ &\int_{\gamma} \nabla f\cdot d\mathbf{r} = f\Big|_{\text{initial}}^{\text{final}}\,, \quad \int_{\mathbf{D}} \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y}\right)dxdy\\ &\qquad = \oint_{\partial\mathbf{D}} (L(x,y)dx + M(x,y)dy) \quad (67)\end{aligned}$$

Further identities can be derived by imposing $\mathbf{F} = \phi(x,y,z)\mathbf{c}$ for the constant vector field $\mathbf{c}$ or similar constraints.

Spherical and Cylindrical coordinates in $\mathbb{R}^3$ are defined in terms of the Cartesian coordinates (x,y,z) as

$$\begin{aligned}z &= r\cos(\theta)\,, \quad y = x\tan(\phi) \quad (68)\\ x &= z\tan(\theta)\cos(\phi), \quad (69)\end{aligned}$$

for the spherical coordinates (r,θ,ϕ) and as

$$x = r\cos(\theta), \ y = r\sin(\theta) \quad (70)$$

for the cylindrical coordinates (r,θ,z).

Polar coordinates in $\mathbb{R}^d$ $(r,\theta_1,\ldots,\theta_{d-1})$ are defined in terms of the Cartesian coordinates $(x_1,\ldots,x_d)$ as

$$\begin{aligned}x_1 &= r\cos(\theta_1)\,, \quad x_d = x_{d-1}\tan(\theta_{d-1}) \quad (71)\\ x_i &= x_{i-1}\tan(\theta_{i-1})\cos(\theta_i) \quad \text{for } 1 < i < d \quad (72)\end{aligned}$$

In two-dimensions, this reduces to the familiar polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$; in 3 dimensions, it becomes the familiar spherical coordinates for $(x_1,x_2,x_3) = (z,x,y)$ and $(\theta_1,\theta_2) = (\theta,\phi)$. In higher dimensions, polar coordinates are also called *hyperspherical coordinates.*

Cylindrical coordinates in $\mathbb{R}^d$ $(r,\theta_1,\ldots,\theta_{n-1},x_n, x_{n+1},\ldots,x_d)$ is a coordinate system such that a subset $\mathbb{R}^n$ of the total space $\mathbb{R}^d$ (for $n < d$) is converted into the polar coordinates. For instance, if we convert $\mathbb{R}^2$ of $\mathbb{R}^3$ into polar coordinates, we obtain the familiar *3d cylindrical coordinates*, i.e. $(x,y,z) = (r\cos\theta, r\sin\theta, z)$.

(Anti)holomorphic function of a complex variable is a function f for which the derivative with respect to z ($\bar{z}$) is uniquely defined, i.e.

$$\begin{aligned}\frac{df(x,y)}{dz} &:= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x + i\Delta y}\\ \left(\frac{df(x,y)}{d\bar{z}}\right. &:= \left.\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x - i\Delta y}\right)\end{aligned} \quad (73)$$

is well-defined and independent of the order of limits (this condition leads to *Cauchy-Riemann equations*). As any antiholomorphic function can be written as *complex conjugate of a holomorphic function*, one usually focuses on the analysis of holomorphic functions alone.

An analytic function is a function expandable as a convergent power series. Cauchy's integral formula ensures that *a complex analytic function* (with a series expansion in z) is equivalent to a *holomorphic function.*

Cauchy's integral formula for a function f complex-analytic in the region $D \subset \mathbb{C}$ can be written as

$$f(z) = \oint_{\partial D} \frac{f(\omega)}{\omega - z} \frac{d\omega}{2\pi i} \tag{74}$$

Laurent series of a function complex analytic for $R_1 < |z-a| < R_2$ is the convergent series expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$.

A pole of a complex analytic function f is a point $a \in \mathbb{C}$ such that $f(a)$ is singular and S is a non-empty set for $S = \{m \in \mathbb{Z} \mid (z-a)^m f(z) \text{ is analytic at } a\}$. $\min(S)$ is called the order of the pole.

A zero of a function f is the value a such that $f(a) = 0$. $\min\{m \in \mathbb{Z} \mid \lim_{z\to a} (z-a)^{-m} f(z) \neq 0\}$ is called order of zero.

A meromorphic function f in a domain D is a holomorphic function in D except a set of points at which f has a pole. For example, $z \to \frac{1}{\sin(z)}$, $z \to \frac{e^z}{z}$ are meromorphic functions in $D = \mathbb{C}$. If we also include infinity ($D = \mathbb{C} \cup \{\infty\}$), they are no longer meromorphic as they are singular at infinity but that singularity is not a pole. In fact, *the only meromorphic functions in $D = \mathbb{C} \cup \{\infty\}$* are rational functions, e.g.

$$z \to \frac{(z-1)(z+i)}{(2z+\pi)(z-i+1)}.$$

Residue of a complex function at an isolated singularity a is defined as

$$\text{Res} : (\mathbb{C} \to \mathbb{C}\,,\ \mathbb{C}) \to \mathbb{C} \tag{75}$$

$$\text{Res}(f, a) = \frac{1}{2\pi i} \oint_{C_a} f(z) dz \tag{76}$$

for an infinitesimal closed contour C_a centered at a.

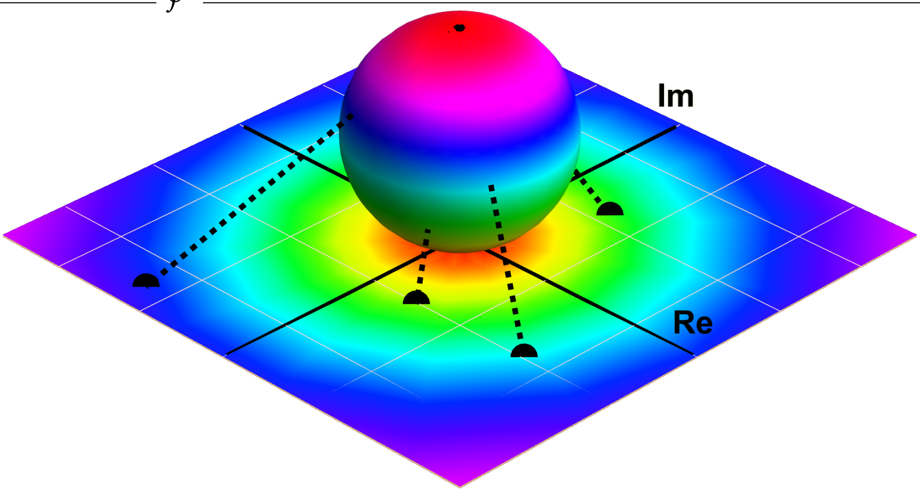
Cauchy's principal value (denoted p.v.) is for our purposes defined via the relation

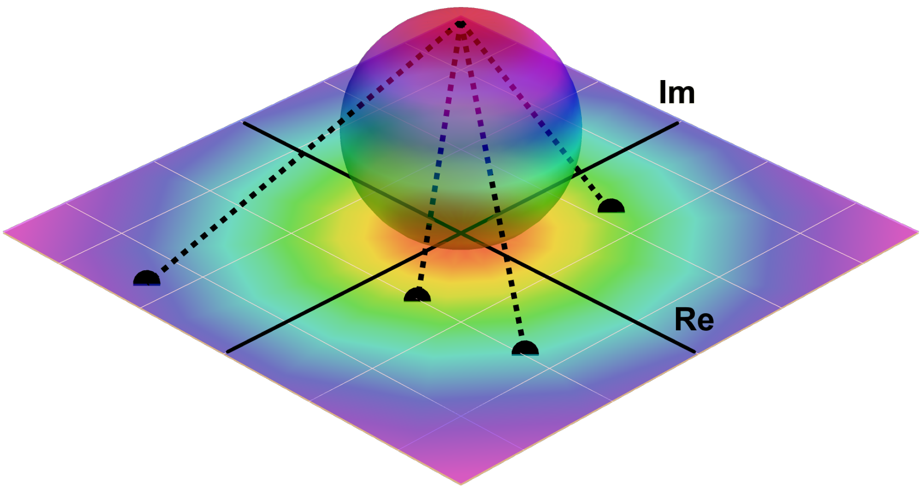
$$\text{p.v.} \int_a^c f(x)dx = \lim_{\epsilon\to 0} \left[\int_a^{b-\epsilon} f(x)dx + \int_{b+\epsilon}^{c} f(x)dx \right] \tag{77}$$

for $a < b < c$, where $f(x)$ is assumed to be analytic in $[a, c] \backslash \{b\}$. If $f(x)$ is analytic at b, the principle value gives the same result with the ordinary integral; on the other hand, if $f(x)$ is not analytic at b, the principle value assigns a well-defined value to the integral which would be otherwise ill-defined as a function.

Conformal transformation (for our purposes) is any mapping $x \to x'$ of the coordinates for which the angles between (co)tangent vectors do not change, e.g. $\frac{\langle dx, dy \rangle}{\sqrt{|dx||dy|}} = \frac{\langle dx', dy' \rangle}{\sqrt{|dx'||dy'|}}$; for instance, translation $x' = x + a$, rotation $x' = e^{i\theta} x$ or scaling $x' = \lambda x$ are so.

Riemann sphere (denoted $\hat{\mathbb{C}}$) is the *compactification* of the complex plane $\mathbb{C}$. More simply, it is the inclusion of *infinity* as a single point to the complex plane, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, such that we get a complete symmetry between large numbers and small numbers: large numbers are points close to the *north pole* (by convention), whereas small numbers are points close to the *south pole*; the map $z \to \frac{1}{z}$ sends such numbers to each other and switches north and south poles (hence $\infty \leftrightarrow 0$).





Stereographic projection is a conformal mapping between $\mathbb{R}^d$ ($d-$dimensional plane) and S^d ($d-$sphere); however, we are only interested in the map between $\mathbb{R}^2$ (the complex plane) and S^2 (the Riemann sphere). Geometrically, the mapping can be readily applied as follows: **(1)** embed S^2 and $\mathbb{R}^2$ into $\mathbb{R}^3$ such that the origin of $\mathbb{R}^2$ and south pole of S^2 coincide; **(2)** draw a line from north pole to a point $z \in \mathbb{R}^2$; **(3)** the intersection of the line with S^2 gets mapped to z.

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2 Fill-in the blanks

Each unambigiously correct answer is worth 3 point respectively.

Question: 1 . . . . *(6 points)*

Consider a function f which is analytic on the whole Riemann sphere except at the lowest positive integer at which it has a single pole. We are also given the information that this function yields five when evaluated at the lowest two-digit prime number, and that it goes to zero as its argument is taken to infinity. If we evaluated the function at three instead, we would then get the number ***twenty five / 25***.

Solution 1.1 *As we are told that the function f is analytic on the whole Riemann sphere except on a set of points where it has poles, it is a rational meromorphic function as explained in "Notations & Conventions" section. Since its only pole is a single pole at the lowest positive integer, we conclude $f(z) = \frac{a}{z-1} + b$ for constants a and b. Since we are further given $\lim_{z\to\infty} f(z) = 0$ and $f(11) = 5$, we fix $f(z) = \frac{50}{z-1}$ which yields $f(3) = 25$.*

Let us analyze a different function, g, which is a rational function with one zero and two simple poles on the complex plane. In other words, the function yields the result zero only by a single element of the complex plane (we call that element "the zero of the function") and $1/g(z)$ is zero only for two specific values for z, which we identify as the simple poles of this function g. We are told that "the zero of the function" is the arithmetic average of the poles, and that g is an odd-function. With these information, we can derive that the product of the poles of g is equal to the number ***thirty six / 36*** if we also know that g evaluated at four yields the value of what g yields when evaluated at nine.

Solution 1.2 *We are told that the function is rational with one zero and two poles, hence it takes the form*

$$g(z) = k\frac{z-a}{(z-b)(z-c)} \tag{78}$$

for some unknowns a, b, c, k. As we are told that the zero is the arithmetic average of the poles, we can write

$$g(z) = \frac{k}{2}\frac{2z-b-c}{(z-b)(z-c)} \tag{79}$$

Since g is also an odd function, we need $g(-z) = -g(z)$, which leads to

$$\frac{k}{2}\frac{2z-b-c}{(z-b)(z-c)} = -\frac{k}{2}\frac{-2z-b-c}{(-z-b)(-z-c)} \tag{80}$$

which simplifies to

$$\frac{2z-b-c}{(z-b)(z-c)} = \frac{2z+b+c}{(z+b)(z+c)} \tag{81}$$

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This needs to be true for all z! We can then evaluate at $z = 0$ to see that it yields $b + c = 0$, hence we get

$$g(z) = k\frac{z}{(z-b)(z+b)} \tag{82}$$

where $b \neq 0$ is needed as we are told that g has two simple poles (for $b = 0$, g has a double pole).

Our last piece of information is the statement "g evaluated at four yields the value of what g yields when evaluated at six", i.e. $g(4) = g(9)$. Thus

$$k\frac{4}{(4-b)(4+b)} = k\frac{9}{(9-b)(9+b)} \tag{83}$$

which means

$$4(9-b)(9+b) = 9(4-b)(4+b) \tag{84}$$

hence $4(81-b^2) = 9(16-b^2)$, meaning $4 \times 81 - 9 \times 16 = -9b^2 + 4b^2$. Thus

$$-5b^2 = 180 \quad \Rightarrow \quad b^2 = -36 \quad \Rightarrow \quad b = \pm 6i \tag{85}$$

Therefore, "the product of the poles of g", is 36.

3 Choose the correct option

You do not need to show your derivation in this part.
Incorrect answer for a question of X point is worth $-X/7$ points: this ensures that the randomly given answer has an expectation value of 0 point.

Question: 2 . . . . *(10½ points)*

In equations (36-43), we set our conventions for various Fourier transforms and their inverses: please refer to these conventions for the calculations below.

In the parts below, choose the correct $g(x, y)$ such that $f(x, y)$ is an analytic function (each correct answer is worth 2.1 points).

(a) For the function $f(x) = \exp\left(-\frac{5+\sqrt{41}}{4}|x|\right)$, find its Fourier transform $\hat{f}(k)$ evaluated at $k = i$.

☐ $4/3$ ☐ $(8\pi)/3$ ☐ $2/(3\pi)$ ☐ π
■ $4/5$ ☐ $(8\pi)/5$ ☐ $2/(5\pi)$ ☐ $1/\pi$

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Solution 2.1 *We are told to work with the given Fourier conventions, hence we know via (37) that for $f(x) = \exp(-m|x|)$ with $m > 0$ we have*

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) = \int_{-\infty}^{\infty} dx e^{-ikx} e^{-m|x|} = \int_{0}^{\infty} dx e^{-(m+ik)x} + \int_{-\infty}^{0} dx e^{(m-ik)x} \tag{86}$$

By changing the dummy variable x to $-x$ in the second integral, we can rewrite it as

$$\hat{f}(k) = \int_{0}^{\infty} dx e^{-(m+ik)x} + \int_{0}^{\infty} dx e^{-(m-ik)x} \tag{87}$$

Both of these integrals are convergent, hence we can immediately write

$$\hat{f}(k) = \left.\frac{e^{-(m+ik)x}}{-(m+ik)}\right|_0^\infty + \left.\frac{e^{-(m-ik)x}}{-(m-ik)}\right|_0^\infty = \frac{1}{m+ik} + \frac{1}{m-ik} = \frac{2m}{m^2+k^2} \tag{88}$$

indicating $\hat{f}(i) = \frac{2m}{m^2-1}$. Since we have $m = \frac{5+\sqrt{41}}{4}$, we see that

$$\hat{f}(i) = \frac{\frac{5+\sqrt{41}}{2}}{\frac{(5+\sqrt{41})^2-16}{16}} = 8\frac{5+\sqrt{41}}{(5+\sqrt{41})^2-16} = 8\frac{5+\sqrt{41}}{50+10\sqrt{41}} = \frac{4}{5} \tag{89}$$

(b) For the function $f(x) = \dfrac{\cos(x)}{1+x^2}$, find among the options below *"the closest approximate value"* for its Fourier transform $\hat{f}(k)$ evaluated at $k = 3$. (*Hint: you may approximate $e \sim \sqrt{7}$ and $\pi \sim 3$.*)

☐ 12/7 ☐ 7/12 ☐ 49/3 ☐ 3/49
☐ 108/49 ☐ 49/108 ☐ 49/12 ■ 12/49

(c) For the function $f(x) = 4\pi i\dfrac{\sin(x)}{1+x^2}$, find among the options below *"the closest approximate value"* for its Fourier transform $\hat{f}(k)$ evaluated at $k = 3$. (*Hint: you may approximate $e \sim \sqrt{7}$ and $\pi \sim 3$.*)

☐ 12/7 ☐ 7/12 ☐ 49/3 ☐ 3/49
■ 108/49 ☐ 49/108 ☐ 49/12 ☐ 12/49

Solution 2.2 *Let's solve the parts **(b)** & **(c)**: We are told to work with the given Fourier conventions, hence*

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we know via (37) that

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \tag{90}$$

thus under Fourier transform, we have the maps

$$\frac{e^{iax}}{1+x^2} \quad \to \quad \int_{-\infty}^{\infty} dx e^{-ikx} \frac{e^{iax}}{1+x^2} = \int_{-\infty}^{\infty} dx \frac{e^{i(a-k)x}}{1+x^2} \tag{91}$$

For $k, a \in \mathbb{R}$ with $k - a \geq 0$, we can close the contour in the lower half plane (LHP) as $\left|e^{i(a-k)x}\right| = \left|e^{(k-a)\,\mathrm{Im}\, x}\right|$ is bounded hence the integrand vanishes fast enough. Thus, by the residue theorem

$$\frac{e^{iax}}{1+x^2} \quad \to \quad -2\pi i \sum_{LHP} \mathrm{res}\left(\frac{e^{i(a-k)x}}{1+x^2}\right) \tag{92}$$

where the minus sign is there because the contour is clockwise (hence negatively oriented). Since the exponential does not have any poles and $\frac{1}{1+x^2} = \frac{1}{(x-i)(x+i)}$ has a single pole at $x = -i$, we conclude

$$\frac{e^{iax}}{1+x^2} \quad \to \quad -2\pi i \left(\frac{e^{i(a-k)x}}{x-i}\right)\Bigg|_{x=-i} \tag{93}$$

hence

$$\frac{e^{iax}}{1+x^2} \quad \to \quad \pi e^{a-k} \tag{94}$$

By the linearity of the Fourier transform, we can immediately conclude

$$\frac{\cos(x)}{1+x^2} = \frac{1}{2}\frac{e^{ix}}{1+x^2} + \frac{1}{2}\frac{e^{-ix}}{1+x^2} \quad \to \quad \frac{1}{2}\pi e^{1-k} + \frac{1}{2}\pi e^{-1-k} \tag{95}$$

and

$$4\pi i \frac{\sin(x)}{1+x^2} = 2\pi \frac{e^{ix}}{1+x^2} - 2\pi \frac{e^{-ix}}{1+x^2} \quad \to \quad 2\pi^2 e^{1-k} - 2\pi^2 e^{-1-k} \tag{96}$$

And for $k = 3$, these become $\left\{\frac{\pi(1+e^{-2})}{2e^2}, \frac{2\pi^2(1-e^{-2})}{e^2}\right\}$ which approximates to $\left\{\frac{12}{49}, \frac{108}{49}\right\}$ when we take $e^2 \sim 7$ and $\pi \sim 3$.

(d) Which one below is the function $f(x)$ evaluated at $x = 1$ if we are given the information $\hat{f}(k) = \frac{1}{(k-i)(k+2i)}$?

- ■ $1/(3e)$
- ☐ $1/(3e^2)$
- ☐ $-1/(3e^2)$
- ☐ $-1/(3e)$
- ☐ $e/3$
- ☐ $e^2/3$
- ☐ $-e^2/3$
- ☐ $-e/3$

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(e) Which one below is equal to $\sum_{n=1}^{\infty} f(n)$ where we know the Fourier transform of $f(x)$ as $\hat{f}(k) = \frac{1}{(k-i)(k+2i)}$?

■ $\frac{1}{3(e-1)}$ ☐ $\frac{1}{3(e^2-1)}$ ☐ $-\frac{1}{3(e^2-1)}$ ☐ $-\frac{1}{3(e-1)}$
☐ $\frac{e-1}{3}$ ☐ $\frac{e^2-1}{3}$ ☐ $-\frac{e^2-1}{3}$ ☐ $-\frac{e-1}{3}$

Solution 2.3 *Let's solve the parts* **(d)** *&* **(e)***: We are told to work with the given Fourier conventions, hence we know via (36) that*

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}(k) \tag{97}$$

leading to

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{(k-i)(k+2i)} \tag{98}$$

In both parts, we need to consider the case $x > 0$: this means $\left|e^{ikx}\right| = \left|e^{-x\,\mathrm{Im}(k)}\right|$ goes to zero around the arc in the upper half plane *(UHP); thus,*

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{(k-i)(k+2i)} = \oint \frac{dk}{2\pi} \frac{e^{ikx}}{(k-i)(k+2i)} = 2\pi i \sum_{UHP} \mathrm{Res}\left(\frac{e^{ikx}}{2\pi(k-i)(k+2i)}\right) \tag{99}$$

Since e^{ikx} does not have any poles in the complex plane and $\frac{e^{ikx}}{2\pi(k-i)(k+2i)}$ has one pole in the upper half plane, at $k = i$, we get

$$f(x) = 2\pi i \left(\frac{e^{ikx}}{2\pi(k+2i)}\right)\Bigg|_{k=i} = \frac{e^{-x}}{3} \tag{100}$$

We then get the answer for part (d): $f(1) = 1/(3e)$. For part (e), we get

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{e^{-n}}{3} = \frac{1}{3}\sum_{n=1}^{\infty}(e^{-1})^n = \frac{1}{3}\sum_{n=0}^{\infty}(e^{-1})^n - \frac{1}{3}(e^{-1})^0 = \frac{1}{3}\frac{1}{1-e^{-1}} - \frac{1}{3} \tag{101}$$

where we used the geometric series in the last equation. We then conclude

$$\sum_{n=1}^{\infty} f(n) = \frac{1}{3}\left(\frac{1}{1-e^{-1}} - 1\right) = \frac{1}{3}\frac{e^{-1}}{1-e^{-1}} = \frac{1}{3(e-1)} \tag{102}$$

Question: 3 . . . . *(17½ points)*

Below, you are given various scalar and vector fields in two or three dimensional Euclidean spaces:

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compute the required quantities in each part (each correct answer is worth 3.5 points)! Note that the standard $2d$ polar, $3d$ spherical, and $3d$ cylindrical coordinates are provided in "Notations & Conventions" section. Likewise, the definition of grad, div, and curl in Cartesian coordinates can be found there!

Hint: one might convert vectors between different coordinates systems either by geometric arguments as you have learnt in your freshman year (also available in course textbooks, for instance see Hildebrand p304, DiPrima p727, Boas p261) or by algebraic arguments as we have discussed in class: please feel free to use whichever is most convenient for you!

Hint 2: you may want to convert the fields to Cartesian coordinates first, then apply differential operators (grad etc.), then convert to other coordinates if needed.

(a) What is $(\hat{x}-\hat{y})\cdot\nabla f$ at $(r,\theta)=\left(\frac{1}{\sqrt{2}},\frac{\pi}{4}\right)$ for the scalar function $f(r,\theta)=r^2\cos(2\theta)$?

☐ -2 ☐ $-\sqrt{2}$ ☐ -1 ☐ 0 ☐ 1 ☐ $\sqrt{2}$ ■ 2 ☐ $2\sqrt{2}$

Solution 3.1 *Let's convert the function to Cartesian coordinates. For that, we can utilize Euler's formula given in the Notations & Conventions section to rewrite* $\cos(2\theta)$ *as*

$$\begin{aligned}\cos(2\theta)=&\frac{e^{2i\theta}+e^{-2i\theta}}{2}=\frac{(\cos(\theta)+i\sin(\theta))^2+(\cos(-\theta)+i\sin(-\theta))^2}{2}\\ =&\frac{(\cos(\theta)+i\sin(\theta))^2+(\cos(\theta)-i\sin(\theta))^2}{2}=\cos(\theta)^2-\sin(\theta)^2\end{aligned}\tag{103}$$

hence

$$f(r,\theta)=r^2(\cos(\theta)^2-\sin(\theta)^2)\quad\rightarrow\quad f(x,y)=x^2-y^2\tag{104}$$

The gradiant in Cartesian coordinates is trivial, i.e.

$$\nabla f=\frac{\partial(x^2-y^2)}{\partial x}\hat{x}+\frac{\partial(x^2-y^2)}{\partial y}\hat{y}=2x\hat{x}-2y\hat{y}\tag{105}$$

Thus

$$(\hat{x}-\hat{y})\cdot\nabla f=(\hat{x}-\hat{y})\cdot(2x\hat{x}-2y\hat{y})=2(x+y)=2r\cos(\theta)+2r\sin(\theta)\tag{106}$$

As $\cos(\pi/4)=\sin(\pi/4)=\frac{1}{\sqrt{2}}$, $(\hat{x}-\hat{y})\cdot\nabla f=2$ *at* $(r,\theta)=\left(\frac{1}{\sqrt{2}},\frac{\pi}{4}\right)$.

(b) Consider $\mathbf{v}(x,y,z)=\left(x^a y^z-yz^{1+b}\right)\hat{x}+\left(xy^{z-1}z-xz^{1-b}\right)\hat{y}+\left(x^c y^z\log(y)-xy\right)\hat{z}$ such that $\nabla\times\mathbf{v}=0$, meaning it is curl-free hence describable with a *scalar potential* $\phi(x,y,z)$ up to a constant vector. What could be $\frac{\partial^3\phi(x,y,z)}{\partial x\partial y\partial z}$ at $(x,y,z)=(7,1,1)$?

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- ☐ -7
- ☐ $1/7$
- ☐ -1
- ☐ 1
- ☐ $-1/7$
- ☐ 7
- ■ 0
- ☐ $2025i\pi$

Solution 3.2 *Since we have $\nabla\times\mathbf{v}=0$, we can conclude $\mathbf{v}=\nabla\phi+\mathbf{c}$ for a constant vector $\mathbf{c}$ and a scalar potential ϕ. From the given form of the $\mathbf{v}$ we can conclude that $\mathbf{c}=0$ which leaves us with the following equation*

$$\nabla\phi=\left(x^ay^z-yz^{1+b}\right)\hat{x}+\left(xy^{z-1}z-xz^{1-b}\right)\hat{y}+\left(x^cy^z\log(y)-xy\right)\hat{z} \tag{107}$$

which means

$$\frac{\partial\phi}{\partial x}=x^ay^z-xz^{1+b}\,,\quad \frac{\partial\phi}{\partial y}=xy^{z-1}z-xz^{1-b}\,,\quad \frac{\partial\phi}{\partial z}=x^cy^z\log(y)-xy \tag{108}$$

First equation can be integrated with respect to x to give

$$\phi(x,y,z)=\frac{x^{a+1}y^z}{a+1}-xyz^{1+b}+f(y,z) \tag{109}$$

for the unknown function f. Differentiating both sides with respect to y and z and comparing with the previous equations then gives

$$\begin{aligned}\frac{x^{a+1}y^{z-1}z}{a+1}-xz^{1+b}+\frac{\partial f(y,z)}{\partial y}&=xy^{z-1}z-xz^{1-b}\\ \frac{x^{a+1}y^z\log(y)}{a+1}-(1+b)xyz^b+\frac{\partial f(y,z)}{\partial z}&=x^cy^z\log(y)-xy\end{aligned} \tag{110}$$

Since this needs to be true for any value of x, we see from first equation that $a=0$ and $b=0$ which then implies in the second equation that $c=1$. Therefore, clearly

$$\mathbf{v}=\nabla(xy^z-xyz) \tag{111}$$

which identically satisfies $\nabla\times\mathbf{v}=0$ (see the first section where the identity $\nabla\times\nabla\phi=0$ is explicitly provided). We then see that

$$\frac{\partial^3\phi(x,y,z)}{\partial x\partial y\partial z}=y^{z-1}+\log(y)zy^{z-1}-1 \tag{112}$$

which gives $\phi(7,1,1)=\log(1)=\log\left(e^{i\pi(2n)}\right)=i\pi(2n)$ for an integer $n\in\mathbb{Z}$. If principle branch is chosen, we need $n=0$; however, for other branches, n can be any integer appropriately. Therefore, $\left.\frac{\partial^3\phi(x,y,z)}{\partial x\partial y\partial z}\right|_{(x,y,z)=(7,1,1)}=i\pi\times\texttt{even integer}$ is a correct answer, and among the options, 0 is the only choice.

The next three questions are about the same three dimensional Euclidean vector field ω which satisfies the following properties in the cylindrical coordinates:

$$\begin{aligned}\nabla\cdot\omega&=\log(r)g'(z)\\ \hat{r}\cdot\omega&=0\end{aligned} \tag{113}$$

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for an unspecified function g.

(c) If we are further given $\hat{y} \cdot \omega = \cos(\theta)^2$, what is $\hat{x} \cdot \omega$ at the point $(r, \theta, z) = \left(3, \frac{5\pi}{8}, -1\right)$?

☐ $-\sqrt{2}$ ☐ $-\frac{\sqrt{2}}{2}$ ☐ $-\frac{\sqrt{4}}{4}$ ☐ 0

■ $\frac{\sqrt{2}}{4}$ ☐ $\frac{\sqrt{2}}{2}$ ☐ $\sqrt{2}$ ☐ ∞

(d) If we are further given $\hat{y} \cdot \omega = \cos(\theta)^2$, what is $\frac{\partial^2 (\hat{z} \cdot \omega)}{\partial z \partial \theta}$?

☐ $r\cos(\theta)$ ☐ $\cos(\theta)$ ■ $\frac{\cos(\theta)}{r}$ ☐ 0

☐ $r\sin(\theta)$ ☐ $\sin(\theta)$ ☐ $\frac{\sin(\theta)}{r}$ ☐ 42

(e) If we are further given $\hat{y} \cdot \omega = 0$, which one below can be true for a certain $g(z)$?

☐ $\hat{r} \cdot (\nabla \times \omega) = \frac{e^z}{r}$ ■ $\hat{\theta} \cdot (\nabla \times \omega) = \frac{e^z}{r}$ ☐ $\hat{r} \cdot (\nabla \times \omega) = \frac{e^z}{\theta}$ ☐ $\hat{\theta} \cdot (\nabla \times \omega) = \frac{e^z}{\theta}$

☐ $\hat{r} \cdot (\nabla \times \omega) = re^z$ ☐ $\hat{\theta} \cdot (\nabla \times \omega) = re^z$ ☐ $\hat{r} \cdot (\nabla \times \omega) = \theta e^z$ ☐ $\hat{\theta} \cdot (\nabla \times \omega) = \theta e^z$

Solution 3.3 *Let us algebraically derive the transformation between the Cylindrical and Cartesian coordinates as we have done in class. Observe that*

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \tag{114}$$

which becomes

$$\frac{\partial}{\partial x} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta} \tag{115}$$

Imposing $\hat{x} = \frac{\partial}{\partial x}$ *and* $\hat{y} = \frac{\partial}{\partial y}$ *then means that we can choose the orthonormal basis as* $\hat{r} = \frac{\partial}{\partial r}$ *and* $\hat{\theta} = \frac{1}{r}\frac{\partial}{\partial \theta}$ *which is guaranteed to be orthonormal as the basis transformation is conducted with an orthogonal matrix* $M^T = M^{-1}$ *such that :*

$$M = \begin{pmatrix} \cos(\theta) & -\sin\theta & 0 \\ \sin\theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = M . \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{pmatrix} \tag{116}$$

Therefore, we summarize the basis change as

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin\theta & 0 \\ \sin\theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{pmatrix}, \quad \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin\theta & 0 \\ -\sin\theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{117}$$

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Let us now construct ω using the given information as

$$\hat{r}\cdot\omega = 0 \quad\Rightarrow\quad \begin{aligned}\omega =&\alpha(x,y,z)\hat{\theta}+\beta(x,y,z)\hat{z}\\ =&-\alpha(x,y,z)\sin(\theta)\hat{x}+\alpha(x,y,z)\cos(\theta)\hat{y}+\beta(x,y,z)\hat{z}\end{aligned} \tag{118}$$

for the unknown functions α, β. We can now immediately tell the answer for the first part, i.e.

$$\hat{y}\cdot\omega = \cos(\theta)^2 \quad\to\quad \alpha(x,y,z)=\cos(\theta) \quad\to\quad \hat{x}\cdot\omega = -\cos(\theta)\sin(\theta) = -\frac{\sin(2\theta)}{2} \tag{119}$$

We can now immediately conclude

$$\hat{x}\cdot\omega\Big|_{r=3,\theta=\frac{5\pi}{8},z=-1} = -\frac{\sin(5\pi/4)}{2} = \frac{\sin(\pi/4)}{2} = \frac{\sqrt{2}}{4} \tag{120}$$

where we also used $\sin(a+\pi) = -\sin(a)$.

Let us move on to the other parts. For this, we need to compute the divergence:

$$\begin{aligned}\nabla\cdot\omega =& -\frac{\partial\alpha(x,y,z)\sin(\theta)}{\partial x}+\frac{\partial\alpha(x,y,z)\cos(\theta)}{\partial y}+\frac{\partial\beta(x,y,z)}{\partial z}\\ =& -\sin(\theta)\frac{\partial\alpha(x,y,z)}{\partial x}+\cos(\theta)\frac{\partial\alpha(x,y,z)}{\partial y}-\alpha(x,y,z)\left(\frac{\partial\sin(\theta)}{\partial x}-\frac{\partial\cos(\theta)}{\partial y}\right)+\frac{\partial\beta(x,y,z)}{\partial z}\\ =& -\sin(\theta)\frac{\partial\alpha(x,y,z)}{\partial x}+\cos(\theta)\frac{\partial\alpha(x,y,z)}{\partial y}-\alpha(x,y,z)\left(\cos(\theta)\frac{\partial\theta}{\partial x}+\sin(\theta)\frac{\partial\theta}{\partial y}\right)+\frac{\partial\beta(x,y,z)}{\partial z}\end{aligned} \tag{121}$$

As $\tan(\theta)=\frac{y}{x}$, *we see that*

$$\begin{aligned}(1+(\tan\theta)^2)\frac{\partial\theta}{\partial x} =& -\frac{y}{x^2} &\to\quad \frac{\partial\theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin(\theta)}{r}\\ (1+(\tan\theta)^2)\frac{\partial\theta}{\partial y} =& \frac{1}{x} &\to\quad \frac{\partial\theta}{\partial y} = \frac{x}{r^2} = \frac{\cos(\theta)}{r}\end{aligned} \tag{122}$$

which means

$$\nabla\cdot\omega = -\sin(\theta)\frac{\partial\alpha(x,y,z)}{\partial x}+\cos(\theta)\frac{\partial\alpha(x,y,z)}{\partial y}+\frac{\partial\beta(x,y,z)}{\partial z} \tag{123}$$

From (115), *we see that this is simply*

$$\nabla\cdot\omega = \frac{1}{r}\frac{\partial\alpha(x,y,z)}{\partial\theta}+\frac{\partial\beta(x,y,z)}{\partial z} \tag{124}$$

hence

$$\frac{1}{r}\frac{\partial\alpha(x,y,z)}{\partial\theta}+\frac{\partial\beta(x,y,z)}{\partial z} = \log(r)g'(z) \tag{125}$$

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We can use this equation to solve β in terms of α; if we rewrite it as

$$\frac{\partial\left(\beta(x,y,z)-\log(r)g(z)\right)}{\partial z}=-\frac{1}{r}\frac{\partial\alpha(x,y,z)}{\partial\theta} \tag{126}$$

we see that the homogeneous solution is $\beta(x,y,z)-\log(r)g(z)=f(x,y)$ for an arbitrary function f whereas the particular solution can be formally written via the fundamental theorem of calculus (integration is anti-derivative for 1d integrals) which leads to

$$\beta(x,y,z)=f(x,y)+\log(r)g(z)-\int_0^z\frac{1}{r}\frac{\partial\alpha(x,y,\tau)}{\partial\theta}d\tau \tag{127}$$

where the choice of lower limit is arbitrary and changing it only amounts to adding an arbitrary constant which is already accounted in the arbitrary function f, hence this is indeed the most general solution, that is

$$\omega=\alpha(x,y,z)\hat{\theta}+\left[f(x,y)+\log(r)g(z)-\int_0^z\frac{1}{r}\frac{\partial\alpha(x,y,\tau)}{\partial\theta}d\tau\right]\hat{z}\quad\text{for}\quad\begin{aligned}\nabla\cdot\omega&=\log(r)g'(z)\\ \hat{r}\cdot\omega&=0\end{aligned} \tag{128}$$

Note that the degree of freedom is consistent with number of constraints. In general, we have three scalar fields, each with 3 degrees of freedom, to describe a vector field ω in $\mathbb{R}^3$ (hence 9 degrees of freedom). By $\hat{r}\cdot\omega=0$, we effectively reduce the vector field to two dimensions (hence 6 degrees of freedom), and the divergence gives a real differential equation which is another constraint, hence we should be left with 5 degrees of freedom: 3 of them are in $\alpha(x,y,z)$ and 2 of them are in $f(x,y)$.

Let us now start solve the second part. We are additionally given $\hat{y}\cdot\omega=\cos(\theta)^2$ which means

$$\alpha(x,y,z)=\cos(\theta) \tag{129}$$

hence

$$\omega=\cos(\theta)\hat{\theta}+\left[f(x,y)+\log(r)g(z)+\frac{\sin(\theta)z}{r}\right]\hat{z}\quad\text{for}\quad\begin{aligned}\nabla\cdot\omega&=\log(r)g'(z)\\ \hat{r}\cdot\omega&=0\\ \hat{y}\cdot\omega&=\cos(\theta)^2\end{aligned} \tag{130}$$

Clearly $\dfrac{\partial^2\left(\hat{z}\cdot\omega\right)}{\partial z\partial\theta}$ is given as

$$\frac{\partial^2}{\partial z\partial\theta}\left[f(x,y)+\log(r)g(z)+\frac{\sin(\theta)z}{r}\right]=\frac{\cos(\theta)}{r} \tag{131}$$

For the last part, we are given $\hat{y}\cdot\omega=0$ which means

$$\alpha(x,y,z)=0 \tag{132}$$

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hence

$$\omega = \left(f(x,y) + \log(r) g(z)\right)\hat{z} \quad \textit{for} \quad \begin{aligned} \nabla\cdot\omega &= \log(r) g'(z) \\ \hat{r}\cdot\omega &= 0 \\ \hat{y}\cdot\omega &= 0 \end{aligned} \tag{133}$$

We can now immediately write down the curl of this field:

$$\begin{aligned} \nabla\times\omega =& \frac{\partial(\hat{z}\cdot\omega)}{\partial y}\hat{x} - \frac{\partial(\hat{z}\cdot\omega)}{\partial x}\hat{y} \\ =& \frac{\partial f(x,y)}{\partial y}\hat{x} - \frac{\partial f(x,y)}{\partial x}\hat{y} + \frac{g(z)}{r}\left(\frac{\partial r}{\partial y}\hat{x} - \frac{\partial r}{\partial x}\hat{y}\right) \end{aligned} \tag{134}$$

where we used $\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{x}{r}$ *and similarly for* y*. From* (117) *we see that*

$$\frac{y\hat{x} - x\hat{y}}{r} = -\hat{\theta} \tag{135}$$

and combining (117) *with* (115) *we see that*

$$\begin{aligned} \hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x} =& \left(\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}\right)\left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial\theta}\right) \\ &- \left(\sin(\theta)\hat{r} + \cos(\theta)\hat{\theta}\right)\left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}\right) \\ =& \hat{r}\frac{1}{r}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{\partial}{\partial r} \end{aligned} \tag{136}$$

hence by rewriting the arbitrary function $f(x,y)$ *as* $f(r,\theta)$ *instead, we conclude*

$$\nabla\times\omega = \hat{r}\frac{1}{r}\frac{\partial f(r,\theta)}{\partial\theta} - \hat{\theta}\left(\frac{g(z)}{r} + \frac{\partial f(r,\theta)}{\partial r}\right) \tag{137}$$

Clearly $\hat{r}\cdot\nabla\times\omega$ *has to be* $z-$*independent. On the contrary,* $\hat{\theta}\cdot\nabla\times\omega$ *can be* $\frac{e^z}{r}$ *for* $g(z) = -e^z$*.*

« « « Congratulations, you have made it to the end! » » »