Phys209: Mathematical Methods in Physics I

Homework 12

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due December 29th 2023, 23:59 TSI.

(1) Problem One

(1 points)

We have discussed in class that any linear ordinary differential equation can be converted to a system of first order differential equations. Let us review this.

(1.1) (1pt)

Consider the differential operator

$$\mathcal{D} :: \Big(\mathbb{C} \to \mathbb{C}\Big) \to \Big(\mathbb{C} \to \mathbb{C}\Big)$$
$$\mathcal{D} = \frac{d^3}{dt^3} + \cos(t)\frac{d^2}{dt^2} + t^3\frac{d}{dt} + 1$$

where 1 at the end stands for the *identity higher order function*, taking any function $f :: \mathbb{C} \to \mathbb{C}$ to itself.

Let $\mathcal{A}$ be the identity $\Big(\mathcal{D} \cdot f\Big)(x) = 0$. How would you express $\mathcal{A}$ as a first-order differential equation of a column matrix?

(1.2) Bonus question

(not graded)

We reviewed in the last lecture that such a first order differential equation can always be formally solved: we first convert it to *Volterra integral equation*, and then solve it iteratively: in the most common usage of this approach in Physics, we obtain the so-called *Dyson series*. Look into these concepts more for your own personal development!

(2) Problem Two

(5 points)

One can also convert linear ordinary differential equations *with constant coefficients* to a system of first-order differential equations, but that rarely makes sense: one can already generically solve such questions by finding the roots through the *characteristic equation*, exponentiating them, and taking their superposition. We discussed this procedure in detail, but let us see how that characteristic equation emerges in this approach.

(2.1) (0.5pt)

Consider an order$-n$ linear ordinary differential equation with constant coefficients:

$$f^{(n)}(x) + a_1 f^{(n-1)}(x) + \cdots + a_n f(x) = 0$$

Show that it can be rewritten in the form

$$\frac{d}{dx}\begin{pmatrix} f(x) \\ f'(x) \\ \vdots \\ f^{(n-2)}(x) \\ f^{(n-1)}(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{pmatrix}\begin{pmatrix} f(x) \\ f'(x) \\ \vdots \\ f^{(n-2)}(x) \\ f^{(n-1)}(x) \end{pmatrix}$$

(2.2) (1pt)

Consider the modified version of the square matrix found in the previous section:

$${}_n\mathcal{M} = \begin{pmatrix} -\lambda & 1 & 0 & \dots & 0 \\ 0 & -\lambda & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 - \lambda \end{pmatrix}$$

which is same matrix as before but the diagonal entries are shifted with $-\lambda$. We would like to compute the determinant of this matrix.

Remember that the determinant is defined as

$$\det\{\}_n\mathcal{M} = \sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2 \dots i_n} \mathcal{M}_{1i_1}\mathcal{M}_{2i_2}\dots\mathcal{M}_{ni_n}$$

for the usual Levi-Civita symbol ϵ. Here, $\mathcal{M}_{ik}$ denotes the entry of the matrix $\mathcal{M}$ at the $i-$th row and $k-$th column.

This definition of the determinant is powerful enough for us to compute it without specifying the value for n. To do that, observe that the only nonzero entries for $\mathcal{M}_{1i_1}$ are $\mathcal{M}_{11} = -\lambda$ and $\mathcal{M}_{12} = 1$, therefore

$$\det\mathcal{M} = -\lambda \sum_{i_2,\dots,i_n} \epsilon_{1 i_2 \dots i_n} \mathcal{M}_{2i_2}\mathcal{M}_{3i_3}\dots\mathcal{M}_{ni_n} + \sum_{i_2,\dots,i_n} \epsilon_{2 i_2 \dots i_n} \mathcal{M}_{2i_2}\mathcal{M}_{3i_3}\dots\mathcal{M}_{ni_n}$$

Show/argue that the second term can be rewritten as

$$\sum_{i_2,\dots,i_n} \epsilon_{2i_2\dots i_n}\mathcal{M}_{2i_2}\mathcal{M}_{3i_3}\dots\mathcal{M}_{ni_n} = -a_n \sum_{i_2,\dots,i_{n-1}} \epsilon_{2i_2\dots i_{n-1}1}\mathcal{M}_{2i_2}\mathcal{M}_{3i_3}\dots\mathcal{M}_{n-1,i_{n-1}}$$

(2.3) (1pt)

By looking at non-zero values of $\mathcal{M}_{ij}$ and available indices in nonzero ϵ, show/argue that

$$\sum_{i_2,\dots,i_{n-1}} \epsilon_{2i_2\dots i_{n-1}1}\mathcal{M}_{2i_2}\mathcal{M}_{3i_3}\dots\mathcal{M}_{n-1,i_{n-1}} = \sum_{i_3,\dots,i_{n-1}} \epsilon_{23i_3\dots i_{n-1}1}\mathcal{M}_{3i_3}\mathcal{M}_{4i_4}\dots\mathcal{M}_{n-1,i_{n-1}}$$

and so on, ultimately leading to

$$\sum_{i_2,\dots,i_{n-1}} \epsilon_{2i_2\dots i_{n-1}1}\mathcal{M}_{2i_2}\mathcal{M}_{3i_3}\dots\mathcal{M}_{n-1,i_{n-1}} = \epsilon_{2,3,4,\dots,n-1,n,1}$$

(2.4) (1pt)

Show/argue that $\epsilon_{2,3,4,\dots,n-1,n,1} = (-1)^{n-1}$, turning our determinant to

$$\det\{\}_n\mathcal{M} = -\lambda \sum_{i_2,\dots,i_n} \epsilon_{1i_2\dots i_n}\mathcal{M}_{2i_2}\mathcal{M}_{3i_3}\dots\mathcal{M}_{ni_n} + (-1)^n a_n$$

(2.5) (1pt)

Show/argue that $\epsilon_{1i_2\dots i_n} = \epsilon_{k_1\dots k_{n-1}}$ for $k_a = i_{a+1} - 1$. Using this information, show/argue that the sum above is actually the determinant for ${}_{n-1}\mathcal{M}$, hence

$$\det({}_n\mathcal{M}) = -\lambda \det({}_{n-1}\mathcal{M}) + (-1)^n a_n$$

(2.6) (0.5pt)

Starting with $\det({}_1\mathcal{M}) = -a_1 - \lambda$, recursively solve for $\det({}_n\mathcal{M})$. You will find that

$$\det({}_n\mathcal{M}) = (-1)^n \left[\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n\right]$$

Therefore, imposing $\det({}_n\mathcal{M}) = 0$ is actually equivalent to writing down the characteristic equation!

Congratulations, you have found yet another derivation of this cute little equation!

(2.7) Bonus question *(not graded)*

The code below constructs the initial square matrix and then computes the determinant of ${}_n\mathcal{M}$ for any value of n:

```
With[{n = 8},
Module[{mat},
mat[n_] :=
Join[Transpose[
Join[{ConstantArray[0, n - 1]},
IdentityMatrix[n - 1]]], {-Reverse@
        Table[Subscript[a, i], {i, n}]}];
Echo[MatrixForm@mat[n]];
Simplify[Det[mat[n] - \\[Lambda] IdentityMatrix[n]]]]]
```